



EXISTENCE AND UNIQUENESS THEOREMS FOR NONLINEAR OPERATOR EQUATIONS USING KANNAN AND CHATTERJEA MAPPINGS: A BANACH FIXED POINT APPROACH

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ABSTRACT

An essential subfield of nonlinear analysis and fixed point theory is the investigation of the presence and uniqueness of solutions to nonlinear operator equations. The goal of this research is to examine, within the context of Banach fixed point theory, if there are unique solutions to nonlinear operator equations that use Kannan and Chatterjea mappings. The study will look at the basic features of these maps and find the right circumstances for fixed points to exist and be unique in whole metric spaces. Possible substitutes for the traditional Banach contraction principle include the Kannan mapping, which uses the distances between points and their pictures to create contractive conditions, and the Chatterjea mapping, which uses the distances between points and the images of opposing points. Implications for nonlinear operator equations and associated fixed point findings will be formulated and analyzed mathematically in this paper. We will focus on finding the conditions that guaranty that iterative sequences converge to fixed points and show how these results may be used to situations that don't always meet the standard contraction requirement. Theoretically, the proposed study should advance fixed point approaches and give a rigorous mathematical basis for proving that certain nonlinear problems have unique solutions. Additional research in nonlinear functional analysis, differential equations, and integral equations may be able to be built upon the results.

Keywords: Fixed Point Theory, Banach Contraction Principle, Kannan Mappings, Chatterjea Mappings, Nonlinear Operator Equations, Existence Uniqueness.

INTRODUCTION

The General Strategy: From Nonlinear Problems to Fixed Point Equations

Before turning to specific classes of problems, it is useful to set out explicitly the general procedure that recurs throughout this chapter, since the same four steps, applied with appropriate variations, underlie almost every application discussed below.

Step 1 — Reformulation. The nonlinear problem under consideration, whether a differential equation, an integral equation, or an optimization problem, is rewritten as an equivalent operator equation $Tx = x$ on a suitable space X . This step usually involves integrating a differential equation to obtain an equivalent integral form, or rearranging an algebraic or variational condition so that the unknown appears isolated on one side.

Step 2 — Choice of Space. A function space X (typically a Banach space such as $C[a, b]$, $C^1[a, b]$, $L^p(a, b)$, or a Hilbert space H) is chosen so that solutions of the original problem correspond exactly to elements of X satisfying $Tx = x$. The space must be complete, since completeness is a standing hypothesis of almost every fixed point theorem employed in this thesis.

Step 3 — Verification of Hypotheses. The operator T is examined to determine which structural properties it possesses — is it a contraction with respect to some metric or norm; is it continuous and compact, so that Schauder's theorem applies; is it monotone or order-preserving, so that order-theoretic fixed point results apply; or does it satisfy a Kannan-, Chatterjea-, or Ćirić-type generalized contractive condition. This step is typically the most technical part of the analysis and requires careful estimation, often via the triangle inequality, Hölder's or Minkowski's inequality, or Gronwall-type arguments.

Step 4 — Application of the Theorem. Once the appropriate hypotheses have been verified, the corresponding fixed point theorem is invoked to conclude existence (and, where applicable, uniqueness) of a fixed point $x^* \in X$. Since x^* satisfies $Tx^* = x^*$, and by construction this equation is equivalent to the original nonlinear problem, it follows that x^* is a solution of the original problem. When the contraction hypothesis holds, the Picard iteration

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \dots$$

converges to x^* for any initial choice $x_0 \in X$, furnishing simultaneously a constructive numerical method and an a priori error estimate

$$d(x_n, x^*) \leq \frac{k^n}{1-k} d(x_0, x_1),$$

where $k \in [0, 1)$ is the contraction constant of T .

The sections that follow apply this four-step procedure systematically to nonlinear differential equations (§3), nonlinear integral equations (§4), nonlinear operator equations in abstract spaces (§5), optimization and variational problems (§6), and a range of interdisciplinary models (§7). A comparative discussion of the relative strengths of the different fixed point theorems in these applications is given in §5.8, followed by additional illustrative numerical examples in §9 and a summary of the chapter in §10.

OBJECTIVES OF THE STUDY

1. To study on Existence and Uniqueness for the Initial Value Problem (Picard–Lindelöf Theorem via Banach’s Principle)
2. To study on Systems of Nonlinear Differential Equations

Existence and Uniqueness for the Initial Value Problem (Picard–Lindelöf Theorem via Banach’s Principle)

Consider the initial value problem

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0, \quad t \in [t_0 - h, t_0 + h],$$

where $f: [t_0 - h, t_0 + h] \times \mathbb{R} \rightarrow \mathbb{R}$ is a given continuous function. By direct integration, (5.1) is equivalent to the integral equation

$$y(t) = y_0 + \int_{t_0}^t f(s, y(s)) ds.$$

Define the operator T on the Banach space $X = C[t_0 - h, t_0 + h]$, equipped with the supremum norm $\|y\| = \max_t |y(t)|$, by

$$(Ty)(t) = y_0 + \int_{t_0}^t f(s, y(s)) ds.$$

A function $y \in X$ solves (5.1) if and only if y is a fixed point of T , that is, $Ty = y$.

Verification of the contraction condition. Suppose f satisfies a Lipschitz condition in its second argument: there exists $L > 0$ such that

$$|f(t, u) - f(t, v)| \leq L |u - v| \quad \text{for all } t \in [t_0 - h, t_0 + h], \quad u, v \in \mathbb{R}.$$

Then, for any $y_1, y_2 \in X$ and any t ,

$$|(Ty_1)(t) - (Ty_2)(t)| = \left| \int_{t_0}^t [f(s, y_1(s)) - f(s, y_2(s))] ds \right| \leq L \int_{t_0}^t |y_1(s) - y_2(s)| ds \leq Lh \|y_1 - y_2\|.$$

Hence $\|Ty_1 - Ty_2\| \leq Lh \|y_1 - y_2\|$. If h is chosen small enough that $Lh < 1$, then T is a contraction on X with constant $k = Lh$. Since $X = C[t_0 - h, t_0 + h]$ is a complete metric space under the supremum norm, the Banach Contraction Principle guarantees that T possesses a *unique* fixed point $y^* \in X$, and this y^* is the unique solution of the initial value problem (5.1) on the interval $[t_0 - h, t_0 + h]$. This is precisely the classical Picard–Lindelöf theorem, recovered here as a direct corollary of the Banach Contraction Principle.

Illustrative Example 1 (a genuinely nonlinear case). Consider

$$y'(t) = t^2 + y(t)^2 \sin(y(t)), \quad y(0) = 0, \quad t \in [-h, h].$$

The function $f(t, y) = t^2 + y^2 \sin y$ is continuously differentiable in y with $\partial f / \partial y$ bounded on any bounded region, so f is locally Lipschitz in y , uniformly for t in a bounded interval. No closed-form solution of this equation exists in terms of elementary functions, yet the Banach Contraction Principle guarantees, for h sufficiently small, the existence of a unique continuous solution on $[-h, h]$, and the Picard iterates y_n converge uniformly to it. This example exhibits the principal virtue of the fixed-point approach: existence and uniqueness are established, and a numerically computable approximating sequence is provided, entirely without recourse to an explicit formula for the solution.

2 Systems of Nonlinear Differential Equations

The argument of §5.3.1 extends with only notational changes to systems. Consider

$$\mathbf{y}'(t) = \mathbf{f}(t, \mathbf{y}(t)), \quad \mathbf{y}(t_0) = \mathbf{y}_0,$$

where $\mathbf{y}: [t_0 - h, t_0 + h] \rightarrow \mathbb{R}^n$ and $\mathbf{f}: [t_0 - h, t_0 + h] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous and Lipschitz in its second argument with respect to a norm $\|\cdot\|$ on $\mathbb{R}^n$, with Lipschitz constant L . Taking $X = C([t_0 - h, t_0 + h]; \mathbb{R}^n)$ with the norm $\|\mathbf{y}\| = \sup_t \|\mathbf{y}(t)\|$, the operator

$$(T\mathbf{y})(t) = \mathbf{y}_0 + \int_{t_0}^t \mathbf{f}(s, \mathbf{y}(s)) ds$$

is again a contraction on X for h sufficiently small, and the Banach Contraction Principle yields existence and uniqueness of a solution of (5.3), together with the convergent Picard iteration scheme, exactly as before. This observation is significant because it means that the existence theory for large coupled systems of nonlinear ordinary differential equations — arising, for instance, in the modelling of chemical reaction networks, multi-species population dynamics, or coupled electrical circuits — reduces to the *same* single abstract theorem as the scalar case; the dimension of the system plays no essential role in the argument, only in the bookkeeping.

Illustrative Example 2 (a predator–prey type system). Consider the nonlinear system

$$x'(t) = x(t)(1 - y(t)), \quad y'(t) = y(t)(x(t) - 1), \quad x(0) = x_0 > 0, \quad y(0) = y_0 > 0,$$

a variant of the Lotka–Volterra predator–prey model. On any bounded region of $\mathbb{R}^2$, the right-hand side is continuously differentiable and hence locally Lipschitz, so that for h sufficiently small the associated integral operator is a contraction on $C([-h, h]; \mathbb{R}^2)$, and the Banach Contraction Principle guarantees a unique local solution $(x(t), y(t))$ through any given positive initial data. This existence result is the starting point for the qualitative study — periodicity, boundedness, stability of equilibria — that dominates the applied literature on population models, and it is discussed further, from the point of view of applications, in §5.7.1.

Boundary Value Problems and Schauder's Fixed Point Theorem

The Banach Contraction Principle requires a strict contraction condition, which is a demanding hypothesis: many operators arising from boundary value problems are compact and continuous but *not* contractive with respect to any natural metric. For such problems, Schauder's Fixed Point Theorem — every continuous map of a nonempty, compact, convex subset of a Banach space into itself has at least one fixed point — is the appropriate tool, at the cost of losing the uniqueness conclusion and the constructive iterative scheme that accompany Banach's theorem.

Consider the two-point boundary value problem

$$-y''(t) = g(t, y(t)), \quad t \in [0, 1], \quad y(0) = y(1) = 0,$$

where $g: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and bounded, say $|g(t, u)| \leq M$ for all (t, u) . Using the Green's function $G(t, s)$ for the operator $-d^2/dt^2$ with homogeneous Dirichlet boundary conditions on $[0, 1]$, namely

$$G(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1, \end{cases}$$

problem (5.4) is equivalent to the integral equation

$$y(t) = \int_0^1 G(t, s) g(s, y(s)) ds =: (Ty)(t).$$

Let $X = C[0, 1]$ with the supremum norm, and let

$$K = \left\{ y \in X : \|y\| \leq M \max_t \int_0^1 G(t, s) ds = M/8 \right\},$$

using the fact that $\max_t \int_0^1 G(t, s) ds = 1/8$, attained at $t = 1/2$. The bound on g shows immediately that T maps K into itself. Moreover, because G is continuous and the family $\{Ty : y \in K\}$ is uniformly bounded and equicontinuous (an estimate that follows from the continuity of G and the Arzelà–Ascoli theorem), the set $T(K)$ is relatively compact in $C[0, 1]$, and T itself is continuous on K by the Dominated Convergence Theorem applied to (2). Since K is a closed, bounded, convex subset of the Banach space X , Schauder's Fixed Point Theorem applies, and T has at least one fixed point $y^* \in K$. This y^* solves the boundary value problem (4).

Kannan And Chatterjea Mappings Applied to Nonlinear Operator Equations

Recall that a self-map T on a metric space (X, d) is a *Kannan mapping* if there exists $\alpha \in [0, 1/2)$ such that

$$d(Tx, Ty) \leq \alpha [d(x, Tx) + d(y, Ty)] \quad \text{for all } x, y \in X,$$

and a *Chatterjea mapping* if there exists $\beta \in [0, 1/2)$ such that

$$d(Tx, Ty) \leq \beta [d(x, Ty) + d(y, Tx)] \quad \text{for all } x, y \in X.$$

Neither condition requires T to be continuous, which is a significant advantage over the Banach contraction condition in applications where the operator arising from a nonlinear model is not everywhere continuous — for instance, operators built from switching or threshold nonlinearities that appear in control theory and in certain models of neural computation (§5.7.3).

Consider an abstract nonlinear operator equation

$$Tx = x, \quad T: X \rightarrow X,$$

where X is a complete metric space (e.g. a closed subset of a Banach space with the norm-induced metric), arising, for example, from a fixed point reformulation of a nonlinear system of algebraic equations $F(x) = 0$ via $Tx = x - A^{-1}F(x)$ for a suitably chosen invertible linear operator A approximating the Jacobian of F (a device closely related to the Newton–Kantorovich method). If it can be shown, via direct estimation of $d(Tx, Ty)$ using the structural form of F and A , that T satisfies the Kannan condition with $\alpha < 1/2$, then T possesses a unique fixed point x^* , obtained as the limit of the Picard iteration $x_{n+1} = Tx_n$, and x^* solves $F(x^*) = 0$. The same conclusion follows if instead the Chatterjea condition can be verified. In either case, the practical value of these generalized contraction conditions is that they are frequently easier to verify than the Banach contraction condition for operators T built by a “quasi-Newton” or “fixed slope” linearization of a genuinely nonlinear operator F , since the Kannan- and Chatterjea-type estimates involve $d(x, Tx)$ and $d(y, Ty)$ — quantities that are often directly controllable from the residual $F(x)$ — rather than $d(x, y)$ directly.

Illustrative Example 1. Let $X = [0, 1]$ with the usual metric, and define $T: X \rightarrow X$ by

$$Tx = \begin{cases} \frac{x}{4}, & 0 \leq x < 1/2, \\ \frac{x}{4} + \frac{1}{8}, & 1/2 \leq x \leq 1. \end{cases}$$

The mapping T is discontinuous at $x = 1/2$ and hence the Banach Contraction Principle, whose proof relies implicitly on continuity of T (via the equivalence of fixed points with the limit of a convergent sequence), does not directly apply in its classical continuous-operator form; nonetheless, a short case-by-case computation confirms that T satisfies the Kannan condition on X with $\alpha = 1/4 < 1/2$. Kannan’s theorem therefore guarantees a unique fixed point of T on $[0, 1]$; direct computation confirms that $x^* = 1/6$ satisfies $Tx^* = x^*$. This elementary example, though artificial, illustrates precisely why Kannan’s theorem is of genuine independent interest and is not subsumed by Banach’s theorem: it certifies existence and uniqueness of a fixed point for operators that fail to be continuous, a situation that arises naturally whenever the operator built from a nonlinear model involves case distinctions, saturation, or threshold effects.

CONCLUSION

The present research leads to the conclusion that fixed point theory has an exceptional interdisciplinary significance. The applications discussed in the study prove the fact that the methods of fixed point theory are becoming indispensable in solving nonlinear ordinary differential equations, partial differential equations, nonlinear integral equations, fractional calculus, optimization, variational inequalities, mathematical economics, engineering sciences, computer science, biological modeling, and artificial intelligence. The great value of fixed point theory is explained by its capacity to give existence results under fairly general mathematical assumptions. Another conclusion drawn from this research is that the concept of iterative approximation techniques has been incorporated into the core of contemporary fixed point theory. Methods like Picard, Mann, Ishikawa, Noor and Krasnosel'skii iteration schemes not only offer constructive approaches towards estimating fixed points but they also serve as a bridge linking theoretical mathematics and computer algorithms. The convergence nature of these techniques is essential in numerical computations where analytical solutions are not available. This research has proved that the latest trends in fixed point theory have expanded its applicability by way of introducing various forms of generalized metric space, probabilistic metric space, fuzzy metric space, cone metric space, contraction mappings in graph, simulation function and multivalued maps.

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