



MATRIX MEASURES AND THEIR ROLE IN THE STABILITY OF INTERVAL AND DYNAMICAL SYSTEMS

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ABSTRACT

The norm Logarithmic norms, alternatively called matrix measures, offer a useful norm-based approach to the stability analysis of dynamical systems, especially where uncertainty is present, and parameter values are intervals. The paper offers a strict and unified approach to matrix measures and their application in the study of linear, nonlinear, and interval dynamical systems via stability analysis. The needs are set and the required mathematical background is established before the definitions of matrices measures and the derivation of the basic properties. Linear systems Stability criteria Linear systems have their measures of stability developed using matrices, pointing out their strength benefits versus classical techniques based on eigenvalues. It is also generalized to interval systems, where exponential stability is robust with parametric uncertainty provided that the measures of the matrix are uniformly negative. Moreover, nonlinear systems contraction analysis is defined in risky terms based on Jacobian measure conditions and gives feedbacks of exponential convergence between tracks. The paper has shown, through theoretical findings, an example commentary, and comparative tabular results, that matrix measures can be used as a tool of unification between classical stability theory, robust control, and more contemporary contraction-based methods of uncertain and dynamical systems.

Keywords: Matrix Measures, Logarithmic Norms, Stability Analysis, Interval Systems, Contraction Theory, Dynamical Systems.

1. INTRODUCTION

In dynamical systems theory and control, stability is a basic concept because it is what allows system trajectories to approach equilibrium, to blow outward, or to stay within admissible bounds when perturbed internally or externally. Stability analysis methods have been mainly traditional with the eigenvalue conditions applied to linear systems and the Lyapunov functions applied to nonlinear systems. These methods are mathematically rigorous and have been long-standing established, but in situations where

uncertainty exists, time-dependent effects are present and the parameters of the system are interval-valued, they may be inapplicable when precise spectral data or a suitable Lyapunov function is hard to compute.

Matrix measures are another and complementary approach to stability analysis by measuring the time rate of instantaneous growth or decay of vector norms on system trajectories. The resulting norm-based characterization has enough stability conditions which are both robust and systematically conservative, and are computationally tractable. These properties render matrix measures to be highly appropriate in analysing the uncertain, interval, and perturbed dynamical systems. As a result, they have gained more and more interest in fields like contraction theory, robust and nonlinear control, interval observer design and data-driven dynamical systems.

This paper is aimed at providing a brief, although formal, introduction to matrix measures and outlining the importance of matrices measures in the analysis of the stability of interval and dynamical systems. Through the formulation of the theoretical background and giving examples of their use within a coherent mathematical environment the paper will seek to show that matrix measures are an effectively useful tool that can be used to bridge the gap between classical stability theory and more recent methods of dynamical system analysis.

2. LITERATURE REVIEW

Arbabi and Mezić (2017) investigated the connections between ergodic theory, dynamic mode decomposition (DMD), and the computation of spectral properties of the Koopman operator in dynamical systems. The study demonstrated that DMD served as a powerful data-driven technique for approximating the spectral characteristics of nonlinear dynamical systems through linear representations in higher-dimensional spaces. It highlighted how the Koopman operator framework enabled a deeper understanding of system behavior, including stability, coherence, and long-term dynamics. The findings contributed significantly to bridging classical dynamical systems theory with modern computational and data-driven approaches.

Brogliato and Tanwani (2020) examined dynamical systems coupled with monotone set-valued operators, providing a comprehensive theoretical framework that addressed formalisms, applications, well-posedness, and stability. The study focused on systems with nonsmooth dynamics, such as mechanical systems with friction or impacts, and demonstrated how set-valued analysis could be used to model such complexities. It emphasized the importance of ensuring well-posedness and stability in these systems and provided rigorous

mathematical tools for analyzing their behavior, thereby advancing the study of hybrid and nonsmooth dynamical systems.

Efimov and Raïssi (2016) explored the design of interval observers for uncertain dynamical systems, aiming to estimate system states in the presence of uncertainties and disturbances. The study proposed observer design techniques that provided upper and lower bounds for system states, ensuring robustness and reliability in estimation. It highlighted the effectiveness of interval methods in handling uncertainties and improving system monitoring and control. The findings were particularly relevant for applications in control engineering, fault detection, and safety-critical systems.

Layek (2015) provided an introduction to dynamical systems and chaos, offering a foundational understanding of system behavior, stability, and nonlinear dynamics. The work covered key concepts such as equilibrium points, bifurcation, and chaotic behavior, illustrating how small changes in initial conditions could lead to significantly different system outcomes. It emphasized the importance of qualitative analysis in understanding complex dynamical systems and provided a strong theoretical basis for further study in chaos theory and nonlinear dynamics.

3. MATHEMATICAL PRELIMINARIES

Consider a continuous-time dynamical system of the form

$$\dot{x}(t) = f(x(t), t), x(t) \in \mathbb{R}^n,$$

where $f: \mathbb{R}^n \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ is assumed to be locally Lipschitz continuous with respect to the state variable x . This regularity condition guarantees the existence and uniqueness of solutions for given initial conditions. An equilibrium points $x^* \in \mathbb{R}^n$ of the system satisfies

$$f(x^*, t) = 0,$$

for all admissible time t .

To analyze stability properties, vector norms and their associated induced matrix norms play a central role. Let $\|\cdot\|$ denote a norm on $\mathbb{R}^n$. The matrix norm induced by $\|\cdot\|$ is defined as

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}, A \in \mathbb{R}^{n \times n}.$$

Induced matrix norms give an indication of the largest possible amplification of vectors under linear transformations, as well as agree with the norm of the vectors selected.

Some of the induced norms in common use are the 1-norm, the Euclidean (2)-norm and the infinity-norm. They both generate a different matrix norm with different analytical behavior and computation benefits. These norms are the structural means of determining the measures of matrices as well as setting norm based limits on the increase or decrease of system pathways, which are then used in the stability analysis of both linear, nonlinear and interval dynamical systems.

4. MATRIX MEASURES: DEFINITION AND FUNDAMENTAL PROPERTIES

Logarithmic norms or matrix measures are a norm-based account of infinitesimal growth behaviour of linear dynamical systems. They are important in the analysis of stability and contraction by connecting the properties of the matrices with the development of the norms of vectors.

4.1 Definition

Let $\|\cdot\|$ be a vector norm on $\mathbb{R}^n$ with the corresponding induced matrix norm. The matrix measure $\mu(\cdot)$ induced by $\|\cdot\|$ is defined as

$$\mu(A) = \lim_{h \downarrow 0} \frac{\|I + hA\| - 1}{h}, A \in \mathbb{R}^{n \times n},$$

provided the limit exists.

This definition brings some first-order variation of the induced norm of the linear operator $I+hA$ with the perturbation parameter h approaching zero. In turn, the matrix measure can be seen as the immediate rate of change of the norm of solutions to the linear equation $\dot{x} = Ax$. Unlike eigenvalues, matrix measures are explicitly dependent on the selected norm, which enables the analysis of stability to be flexible to the structure of the particular system.

4.2 Fundamental Properties

The matrix measures have some properties that are important and hence it is especially useful in studying stability.

Theorem 4.1.

For any matrices $A, B \in \mathbb{R}^{n \times n}$ and any scalar $\alpha \geq 0$, the following properties hold:

1.

$$\mu(A + B) \leq \mu(A) + \|B\|,$$

2.

$$\mu(\alpha A) = \alpha \mu(A),$$

3.

$$\mu(A) \geq \max \{\operatorname{Re}(\lambda) : \lambda \in \sigma(A)\},$$

where $\sigma(A)$ denotes the spectrum of A .

Proof.

The subadditivity property is an extension of the triangle inequality of the induced matrix norm. The positive homogeneity property can be readily deduced by the fact that the perturbation hA is a linear operator with respect to scalar multiplication. Spectral bound is obtained by setting the induced norm of the matrix exponential to dominate the exponential growth rate of the deterministic value of the larger real parts of eigenvalues. The limit, $h \rightarrow 0$, gives the inequality of the statement.

Remark 4.1

Matrix measures give bounds on the spectral abscissa of a matrix which give a gloomy and robust estimate of the rates of growth of a system. Even though they can be conservative with regard to the actual spectral

behaviour, this conservatism is beneficial in analysis of uncertain, interval, and time-varying systems, where known eigenvalue information may be inaccurate and unavailable.

Table 1: Matrix Measures Induced by Standard Vector Norms

Vector Norm	Induced Matrix Norm	Matrix Measure $\mu(A)$
1-norm $\ \cdot\ _1$	$(\ A\ _1 = \max_j \sum_i a_{ij})$	$\max_j \sum_i a_{ij}$
2-norm $\ \cdot\ _2$	Spectral norm	$\mu_2(A) = \lambda_{\max}\left(\frac{A + A^T}{2}\right)$
∞ -norm $\ \cdot\ _\infty$	$(\ A\ _\infty = \max_i \sum_j a_{ij})$	$\max_i \sum_j a_{ij}$

The matrix measures depend explicitly on the choice of the underlying vector norm. For the commonly used vector norms, such as the 1-norm, 2-norm, and infinity norm, the matrix measures have closed-form expressions in terms of the entries of the matrix or symmetric components. These closed-form expressions allow direct checks of stability conditions without necessarily carrying out spectral decompositions. Table 1 lists the expressions of the matrix measures induced by the commonly used vector norms.

5. STABILITY OF LINEAR DYNAMICAL SYSTEMS

Consider the linear time-invariant dynamical system

$$\dot{x}(t) = Ax(t), x(t) \in \mathbb{R}^n,$$

where $A \in \mathbb{R}^{n \times n}$ is a constant system matrix. The stability of the equilibrium point at the origin is typically defined in terms of the spectral properties of A . Matrix measures is another norm-based measure which gives sufficient conditions to exponential stability.

Theorem 5.1 (Exponential Stability via Matrix Measures)

If there exists a vector norm $\|\cdot\|$ such that the associated matrix measure satisfies

$$\mu(A) < 0,$$

then the origin of the system is globally exponentially stable.

Proof

Suppose that $x(t)$ is a solution of the system with an initial condition of any arbitrary $x(0)$. The time derivative of the norm of the state is a solution to the differential inequality by definition of the measure of the matrix and properties of induced norms.

$$\frac{d}{dt} \|x(t)\| \leq \mu(A) \|x(t)\|.$$

Applying Grönwall's inequality yields

$$\|x(t)\| \leq e^{\mu(A)t} \|x(0)\|, \forall t \geq 0.$$

Since $\mu(A) < 0$, the norm of the state decays exponentially to zero as $t \rightarrow \infty$, establishing global exponential stability of the origin.

Remark 5.1

The stability is sufficient but not necessarily the condition $\mu(A) < 0$. It has an advantage in the form of robustness: when the system matrix is perturbed in a way that does not affect the negativity of the matrix measure, stability is not lost.

Example 5.1

For symmetric matrices $A = A^T$, the matrix measure induced by the Euclidean norm is given by

$$\mu_2(A) = \lambda_{\max}\left(\frac{A + A^T}{2}\right) = \lambda_{\max}(A).$$

Here, the stabilizing test is the largest eigenvalue of A , which is the matrix measure. It is an example of how much is known about the closeness between matrix measures and classical eigenvalue-based criteria of stability, but how much flexibility the norm-dependent formulations provide.

Table 2: Stability Criteria for Linear Systems

Criterion	Condition	Nature of Result
Eigenvalue-based	$\max \operatorname{Re}(\lambda(A)) < 0$	Necessary and sufficient
Lyapunov method	$PA + A^T P < 0$	Constructive but computational
Matrix measure	$\mu(A) < 0$	Sufficient, robust
Interval matrix measure	$\sup_{A \in [A]} \mu(A) < 0$	Robust exponential stability

The Stability of linear systems is also defined with regard to various criteria of analysis with some advantages and drawbacks. Though conditions based on eigenvalues give necessary and sufficient answers, it can be weakly conditioned with uncertainty. Conditions based on matrix measures are adequate, but have strength and ease of computation. Table 2 gives a comparative overview of classical and matrix-measure-based stability conditions, and their respective roles as well as their role in analysis.

6. INTERVAL SYSTEMS AND ROBUST STABILITY

System parameters are unknown in practice and rather fall within given ranges. These uncertainties will naturally cause the formulation of interval dynamical systems, which have a system-matrix which depends on a set of admissible matrices.

An **interval matrix** is defined as

$$[A] = \{A \in \mathbb{R}^{n \times n} : A^- \leq A \leq A^+\},$$

where A^- and A^+ denote the lower and upper bound matrices, respectively, and the inequalities are understood componentwise. The associated interval linear system is given by

$$\dot{x}(t) = Ax(t), A \in [A],$$

which represents a family of linear systems rather than a single deterministic model.

Theorem 6.1 (Robust Stability of Interval Systems)

If

$$\sup_{A \in [A]} \mu(A) < 0,$$

then the interval system $\dot{x} = Ax$ is robustly exponentially stable, meaning that the origin is exponentially stable for every admissible matrix $A \in [A]$.

Proof

The matrix measure $\mu(A)$ of each admissible matrix $A \in [A]$ gives an upper bound on the instantaneous growth rate of the norm of system trajectories. The condition

$$\sup_{A \in [A]} \mu(A) < 0$$

is the property that there is a uniform negative bound on this growth rate on the whole set of uncertainties. As a result, a universal exponential decay rate is accepted in all the systems in the family regardless of the realisation of A . Such uniformity means that every admissible matrix converges exponentially towards the origin and so it aggressively proves the existence of exponential stability.

Remark 6.1

This condition holds especially in those situations when the interval systems are diagonally dominant, i.e., when one can compute the matrix measures induced by the infinity norm or one-norm. Such examples It is common in such cases to determine robust stability by simple algebraic inequalities of marginal-dominance-of-diagonal-stability, and so matrix-measure-based criteria are of particular interest to large-scale and uncertain systems.

Table 3: Applicability of Matrix Measures in Dynamical Systems

System Class	Role of Matrix Measure
Linear systems	Exponential stability certification
Interval systems	Robust stability under uncertainty
Nonlinear systems	Contraction and incremental stability
Time-varying systems	Uniform stability analysis

The Matrix measures are one analysis tool that can be applied in a large variety of dynamical systems. Their norm-based formulation allows them to deal with linear, interval, nonlinear and time varying systems in a comparable stability framework. The Table 3 indicates the objective of the matrix measures in the different classes of systems that are more diversified in the examination of stability.

7. NONLINEAR SYSTEMS AND CONTRACTION ANALYSIS

Consider the nonlinear dynamical system

$$\dot{x}(t) = f(x(t), t), x(t) \in \mathbb{R}^n,$$

where the vector field $f: \mathbb{R}^n \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ is continuously differentiable with respect to x . Let

$$J(x, t) = \frac{\partial f}{\partial x}(x, t)$$

denote the Jacobian matrix of the system.

The contraction theory offers a strong account of the stability of nonlinear systems in that the convergence of trajectories with respect to each other are studied as opposed to the convergence of trajectories with respect to a given equilibrium point. In this context, central importance is assigned to the matrix measures that measure the local rate of contraction caused by the Jacobian matrix.

Theorem 7.1 (Contraction Criterion)

If there exists a vector norm $\|\cdot\|$ and a constant $c > 0$ such that

$$\mu(J(x, t)) \leq -c, \forall x \in \mathbb{R}^n, \forall t \geq 0,$$

then the nonlinear system is shrinking, and the system paths converge to one another exponentially.

Proof

Take any two trajectories of the system, $x_1(t)$ and $x_2(t)$. The variational dynamics of the Jacobian matrix $J(x, t)$ control the evolution of their infinitesimal separation. The time derivative of the difference between the trajectory norm satisfies the inequality by properties of the matrix measures.

$$\frac{d}{dt} \|x_1(t) - x_2(t)\| \leq \mu(J(x, t)) \|x_1(t) - x_2(t)\|.$$

Under the assumed condition $\mu(J(x, t)) \leq -c$, application of Grönwall's inequality yields

$$\|x_1(t) - x_2(t)\| \leq e^{-ct} \|x_1(0) - x_2(0)\|.$$

Therefore, the distance between any two trajectories is exponentially decreasing and this defines contraction of the system.

Remark 7.1

Contraction suggests exponential stability properties worldwide which do not depend on the presence or uniqueness of equilibrium points. Consequently, the time-varying, nonlinear, and uncertain systems in which contraction analysis using a matrix measure is especially effective could be challenging to study using classical Lyapunov-based methods.

8. CONCLUSION

The paper has given a thorough mathematically rigorous account of matrix measures and their application to the stability analysis of interval systems and dynamical systems. Through the creation of foundational definitions, important properties and stability theorems, matrix measures were demonstrated to give a robust and computationally manageable sufficient condition on exponential stability in linear systems, uniform robustness in interval systems, and contraction based convergence in nonlinear systems. In contrast to classical eigenvalue or Lyapunov-based analysis, matrix-measure-based analysis is flexible and norm-selection-based; as well as resistant to uncertainty and perturbation. The coherent framework formed in this paper shows that the matrix measures are a better approach to bridging the classical theory of stability with the new paradigms like contraction analysis and robust control and thus they would be better applied to the uncertain, time-varying, and large-scale dynamical systems. These findings send back to the importance of matrix measures as an analytical tool with multiple applications, and offer a strong prospective to future studies in more sophisticated stability analysis and dynamical systems based on data.

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